STEP Mark Schemes 2016

Mathematics

STEP 9465/9470/9475

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Introduction

These mark schemes are published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by the Examiners and shows the main valid approaches to each question. It is recognised that there may be other approaches and if a different approach was taken in the exam these were marked accordingly after discussion by the marking team. These adaptations are not recorded here.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

The Admissions Testing Service will not enter into any discussion or correspondence in connection with this mark scheme.

1. (i)

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{x^2 + 2ax + b} dx$$

$$x + a = \sqrt{b - a^2} \tan u$$

$$\frac{dx}{du} = \sqrt{b - a^2} \sec^2 u$$

M1 A1

$$I_1 = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b - a^2} \sec^2 u}{(b - a^2) \tan^2 u + (b - a^2)} du = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b - a^2} \sec^2 u}{(b - a^2) \sec^2 u} du$$

M1 A1 **M1**

$$I_1 = \frac{1}{\sqrt{b - a^2}} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 1\, du = \frac{\pi}{\sqrt{b - a^2}}$$

A1 * **(6)**

(ii)

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2ax + b)^n} dx = \left[\frac{x}{(x^2 + 2ax + b)^n}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{-2nx(x + a)}{(x^2 + 2ax + b)^{n+1}} dx$$

M1 A1

$$I_n = 2n \int_{-\infty}^{\infty} \frac{x^2 + ax}{(x^2 + 2ax + b)^{n+1}} dx = 2n \int_{-\infty}^{\infty} \frac{x^2 + 2ax + b}{(x^2 + 2ax + b)^{n+1}} - \frac{ax + b}{(x^2 + 2ax + b)^{n+1}} dx$$

M1 A1

$$I_n = 2nI_n - 2n \int_{-\infty}^{\infty} \frac{\frac{a}{2}(2x + 2a)}{(x^2 + 2ax + b)^{n+1}} + \frac{(b - a^2)}{(x^2 + 2ax + b)^{n+1}} dx$$

M1

$$I_n = 2nI_n - 2n \left[\frac{\frac{-a}{2n}}{(x^2 + 2ax + b)^n}\right]_{-\infty}^{\infty} - 2n(b - a^2)I_{n+1}$$

A1

$$2n(b - a^2)I_{n+1} = (2n - 1)I_n$$

A1 * (7)

(iii) Suppose $I_k = \frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}} \binom{2k - 2}{k - 1}$ for some integer k, $k \geq 1$

B1

Then $I_{k+1} = \frac{2k-1}{2k(b-a^2)} \frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}} \binom{2k - 2}{k - 1} = \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \times \frac{2(2k-1)}{k} \binom{2k - 2}{k - 1}$

M1

$$\frac{2(2k-1)}{k}\binom{2k-2}{k-1} = \frac{2(2k-1)}{k}\frac{(2k-2)!}{(k-1)!\,(k-1)!} = \frac{2k(2k-1)}{kk}\frac{(2k-2)!}{(k-1)!\,(k-1)!} = \binom{2k}{k}$$

so result true for k+1. **M1 A1**

For $n = 1$,

$$\frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}}\binom{2n-2}{n-1} = \frac{\pi}{(b-a^2)^{\frac{1}{2}}}\binom{0}{0} = \frac{\pi}{(b-a^2)^{\frac{1}{2}}}$$

M1 A1

which is the correct result.

So result has been proved by (principle of)(mathematical) induction. **dB1 (7)**

2. (i) $x = at^2 \Rightarrow \frac{dx}{dt} = 2at$

$y = 2at \Rightarrow \frac{dy}{dt} = 2a$

So $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Thus, the gradient of the normal at *Q* is $-q$. **M1 A1**

But this normal is the chord *PQ* and so has gradient $\frac{2ap-2aq}{ap^2-aq^2} = \frac{2a(p-q)}{a(p-q)(p+q)} = \frac{2}{p+q}$ **M1 A1**

So $-q = \frac{2}{p+q}$ which rearranges to $q^2 + qp + 2 = 0$ **A1* (5)**

(ii) Similarly, $r^2 + rp + 2 = 0$

B1

Making *p* the subject of each result, $p = \frac{-(2+q^2)}{q} = \frac{-(2+r^2)}{r}$

So $2r - 2q + q^2r - qr^2 = 0$

$2(r-q) - qr(r-q) = 0$

As $(r-q) \neq 0$, $qr = 2$ **M1 A1**

The line *QR* is $\frac{y-2aq}{x-aq^2} = \frac{y-2ar}{x-ar^2}$

That is $xy - 2aqx - ar^2y + 2a^2qr^2 = xy - 2arx - aq^2y + 2a^2q^2r$

$2ax(r-q) - a(r^2-q^2)y + 2a^2qr(r-q) = 0$

Again, as $(r-q) \neq 0$, and $\neq 0$, $2x - (r+q)y + 2aqr = 0$ **M1 A1**

Because $qr = 2$, *QR* is $2x - (r+q)y + 4a = 0$

So when $y = 0$, $x = -2a$ and thus $(-2a, 0)$ is a suitable fixed point. **B1 (6)**

(iii) Because $q^2 + qp + 2 = 0$ and $r^2 + rp + 2 = 0$ subtracting gives

$q^2 - r^2 + qp - rp = 0$

Again, as $(r-q) \neq 0$, $q + r + p = 0$ **M1 A1**

So the line *QR* is $2x + py + 4a = 0$ **M1**

The line *OP* is $y = \frac{2ap}{ap^2}x$ i.e. $y = \frac{2}{p}x$ **B1**

Thus the intersection of QR and OP is at $\left(-a, \frac{-2a}{p}\right)$, which lies on $x = -a$. **B1 (5)**

$\frac{-2}{p} = \frac{2q}{2+q^2} = \frac{2r}{2+r^2}$

Suppose $k = \frac{-2}{p}$, then $kq^2 - 2q + 2k = 0$ **M1**

As this equation has two distinct real roots, *q* and *r*, then the discriminant is positive **M1**

and so

$4-8k^2>0$ **A1** so $k^2<\frac{1}{2}$ that is $-\frac{1}{\sqrt{2}}<k<\frac{1}{\sqrt{2}}$ which means that the distance of that point of intersection is less than $\frac{a}{\sqrt{2}}$ from the x axis. **A1* (4)**

3. (i)

$$\frac{d}{dx}\left(\frac{Pe^x}{Q}\right) = \frac{Q(Pe^x + P'e^x) - Pe^xQ'}{Q^2}$$

It is required that $\frac{d}{dx}\left(\frac{Pe^x}{Q}\right) = \frac{x^3-2}{(x+1)^2}e^x$ so it follows that

$$[Q(Pe^x + P'e^x) - Pe^xQ'](x+1)^2 = (x^3-2)e^xQ^2$$

M1 A1

Thus,

$$[Q(P + P') - PQ'](x+1)^2 = (x^3-2)Q^2$$

Letting $x = -1$, $0 = -3[Q(-1)]^2$ so $Q(-1) = 0$ and thus *Q* has a factor $(x+1)$ as required.

M1 A1 (4)

Suppose that the degree of $P(x)$ is *p* and that of $Q(x)$ is q. **M1**

Then the degree of $P'(x)$ is $p-1$ and of $Q'(x)$ is $q-1$
So $P + P'$ has degree *p* , $Q(P+P')$ has degree $p+q$, PQ' has degree $p+q-1$,
$[Q(P+P') - PQ']$ has degree $p+q$, and thus $[Q(P+P') - PQ'](x+1)^2$ has degree $p+q+2$

A1

$(x^3-2)Q^2$ has degree $2q+3$

Thus $p+q+2 = 2q+3$ which means that $p = q+1$ as required. **A1 (3)**

If $Q(x) = x+1$, $Q'(x) = 1$, and so

$$(x+1)(P+P') - P = (x^3-2)$$

M1 A1

That is $xP + (x+1)P' = (x^3-2)$

$P(x) = ax^2 + bx + c$ and so $P'(x) = 2ax + b$ **B1**

Therefore $x(ax^2 + bx + c) + (x+1)(2ax+b) = (x^3-2)$ **M1**

and equating coefficients

$$a = 1\,, b + 2a = 0, c + 2a + b = 0, b = -2$$

A1

These equations are consistent, with $a = 1, b = -2, c = 0$ so $P(x) = x^2 - 2x$ **A1 (6)**

(ii) For such *P* and *Q* to exist, $\frac{d}{dx}\left(\frac{Pe^x}{Q}\right) = \frac{1}{x+1}e^x$

and so

$$[Q(Pe^x + P'e^x) - Pe^xQ'](x+1) = e^xQ^2$$

M1

and

$$[Q(P + P') - PQ'](x+1) = Q^2$$

A1

Letting $x = -1$, $0 = [Q(-1)]^2$ so $Q(-1) = 0$ and thus *Q* has a factor $(x+1)$ as before in (i).

However, letting $Q(x) = (x+1)R(x)$, then **M1**

$$[(x+1)R(P + P') - P(R + (x+1)R')](x+1) = (x+1)^2R^2$$

and so

$$(x+1)(RP + RP' - PR') - PR = (x+1)R^2$$

Letting $x = -1$, $P(-1)R(-1) = 0$, but $P(-1) \neq 0$ as *P* and *Q* have no common factors, and so

$R(-1) = 0$ which means that *R* in turn has a factor $(x+1)$. **A1**

Thus Q must have a factor of $(x+1)^2$.

Suppose $Q(x) = (x+1)^nS(x)$, where $n \geq 2$ and $S(-1) \neq 0$ **M1**

Then

$$[Q(P + P') - PQ'](x+1) = Q^2$$

becomes

$$(x+1)[(x+1)^nS(P + P') - P(n(x+1)^{n-1}S + (x+1)^nS')] = (x+1)^{2n}S^2$$

Dividing by the factor $(x+1)^n$ gives,

$$[(x+1)S(P + P') - P(nS + (x+1)S')] = (x+1)^nS^2$$

A1

Letting $x = -1$, $nP(-1)S(-1) = 0$, but $n \neq 0$, $P(-1) \neq 0$ and $S(-1) \neq 0$ giving a contradiction and hence no such *P* and *Q* can exist. **E1**

4. (i)

$$\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}=\frac{(1+x^{r+1})-(1+x^r)}{(1+x^r)(1+x^{r+1})}=\frac{(x-1)x^r}{(1+x^r)(1+x^{r+1})}$$

B1

Therefore

$$\sum_{r=1}^{N}\frac{x^r}{(1+x^r)(1+x^{r+1})}=\frac{1}{(x-1)}\sum_{r=1}^{N}\left(\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}\right)=\frac{1}{(x-1)}\left[\frac{1}{1+x}-\frac{1}{1+x^{N+1}}\right]$$

M1 **M1 A1**

As $N\to\infty$, as $|x|<1$, $\frac{1}{1+x^{N+1}}\to 1$ **M1**

So

$$\sum_{r=1}^{\infty}\frac{x^r}{(1+x^r)(1+x^{r+1})}=\frac{1}{(x-1)}\left[\frac{1}{1+x}-1\right]=\frac{1}{(x-1)}\left[\frac{1-1-x}{1+x}\right]=\frac{-x}{x^2-1}=\frac{x}{1-x^2}$$

A1 * (6)

(ii)

$$\operatorname{sech}(ry)=\frac{1}{\cosh(ry)}=\frac{2}{e^{ry}+e^{-ry}}=\frac{2e^{-ry}}{1+e^{-2ry}}$$

$$\operatorname{sech}\big((r+1)y\big)=\frac{2e^{-(r+1)y}}{1+e^{-2(r+1)y}}$$

M1

Thus

$$\operatorname{sech}(ry)\operatorname{sech}\big((r+1)y\big)=\frac{4e^{-y}e^{-2ry}}{(1+e^{-2ry})(1+e^{-2(r+1)y})}$$

A1

So if $x=e^{-2y}$, **M1**

$$\sum_{r=1}^{\infty}\operatorname{sech}(ry)\operatorname{sech}\big((r+1)y\big)=4e^{-y}\sum_{r=1}^{\infty}\frac{x^r}{(1+x^r)(1+x^{r+1})}=4e^{-y}\frac{x}{1-x^2}$$

A1 **A1**

Thus

$$\sum_{r=1}^{\infty}\operatorname{sech}(ry)\operatorname{sech}\big((r+1)y\big)=4e^{-y}\frac{e^{-2y}}{1-e^{-4y}}=2e^{-y}\frac{2}{e^{2y}-e^{-2y}}$$

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}\big((r+1)y\big) = 2e^{-y}\operatorname{csch}(2y)$$

M1 A1* (7)

(iii)

$$\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry)\operatorname{sech}\big((r+1)y\big) = 2\left[\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}\big((r+1)y\big) + \operatorname{sech} y\right]$$

M1 A1

$$= 2[2e^{-y}\operatorname{csch}(2y) + \operatorname{sech} y] = 2\left[\frac{2e^{-y}}{\sinh 2y} + \frac{1}{\cosh y}\right] = 2\left[\frac{e^{-y}}{\sinh y \cosh y} + \frac{1}{\cosh y}\right]$$

A1 **M1 A1**

$$= \frac{2}{\cosh y}\left[\frac{2e^{-y} + e^{y} - e^{-y}}{2\sinh y}\right] = \frac{2}{\cosh y}\left[\frac{2\cosh y}{2\sinh y}\right] = 2\operatorname{csch} y$$

M1 **A1 (7)**

5. (i)

$$(1+x)^{2m+1} = 1 + \binom{2m+1}{1}x + \cdots + \binom{2m+1}{m}x^m + \binom{2m+1}{m+1}x^{m+1} + \cdots + x^{2m+1}$$

B1

$$= 1 + \binom{2m+1}{1}x + \cdots + \binom{2m+1}{m}x^m + \binom{2m+1}{m}x^{m+1} + \cdots + x^{2m+1}$$

M1

$$x = 1 \Rightarrow 2^{2m+1} = 2\left[1 + \binom{2m+1}{1} + \cdots + \binom{2m+1}{m}\right] > 2\binom{2m+1}{m}$$

M1

and hence $\binom{2m+1}{m} < 2^{2m}$ **A1* (4)**

(ii) $\binom{2m+1}{m} = \frac{(2m+1)!}{(m+1)!m!}$ is an integer. **E1**

If *p* is a prime greater than $m+1$ and less than or equal to $2m+1$, then *p* is a factor of $(2m+1)!$

E1

and is not a factor of $(m+1)!\,m!$, **E1** and so it is a factor of $\binom{2m+1}{m}$. **E1**

Therefore, $P_{m+1,2m+1}$, which is the product of such primes, divides $\binom{2m+1}{m}$. **E1**

Hence, $kP_{m+1,2m+1} = \binom{2m+1}{m}$ where $k \geq 1$ is an integer, **M1** and hence

$P_{m+1,2m+1} = \frac{1}{k}\binom{2m+1}{m} < \frac{1}{k}2^{2m}$, i.e. $P_{m+1,2m+1} < 2^{2m}$ **A1* (7)**

(iii) $P_{1,2m+1} = P_{1,m+1}P_{m+1,2m+1}$ **M1**

$m \geq 1 \Rightarrow m + m \geq m+1$ i.e. $m+1 \leq 2m$ and so $P_{1,m+1} < 4^{m+1}$ applying given condition **E1**

By (ii), $P_{m+1,2m+1} < 2^{2m} = 4^m$ **M1**

Thus, $P_{1,2m+1} < 4^{m+1}4^m = 4^{2m+1}$ as required. **A1* (4)**

(iv) Suppose $P_{1,m} < 4^m$ for all $m \leq k$ for some particular $k \geq 2$. **E1**

Then if $k = 2m$, $P_{1,k+1} < 4^{k+1}$ by (iii). **E1**

$P_{1,2m+2} = P_{1,2m+1} < 4^{2m+1} < 4^{2m+2}$ (equality as $2m+2$ is not prime) using (iii). **E1**

So if $k = 2m+1$, $P_{1,k+1} < 4^{k+1}$. **E1**

$P_{1,2} = 2 < 4^2$ and hence required result is true by principle of mathematical induction. **dE1 (5)**

6.

$$R\cosh(x+\gamma) = R(\cosh x \cosh\gamma + \sinh x \sinh\gamma)$$

So we require $\mathrm{A} = \mathrm{R}\sinh\gamma$ and $\mathrm{B} = \mathrm{R}\cosh\gamma$ which is possible if $B > A > 0$

Thus $R = \sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$. **B1**

If $B = A$, then $A\sinh x + B\cosh x = Ae^x$ **B1**

If $-A < B < A$, then $A\sinh x + B\cosh x$ can be written

$\mathrm{R}\sinh(x+\gamma) = R(\sinh x\cosh\gamma + \cosh x \sinh\gamma)$

requiring $\mathrm{A} = \mathrm{R}\cosh\gamma$ and $\mathrm{B} = \mathrm{R}\sinh\gamma$.

So $R = \sqrt{A^2 - B^2}$ and $\gamma = \tanh^{-1}\frac{B}{A}$ **B1**

If $B = -A$, then $A\sinh x + B\cosh x = -Ae^{-x}$ **B1**

IF $B < -A$, then $A\sinh x + B\cosh x$ can be written $R\cosh(x+\gamma)$-

requiring $\mathrm{A} = \mathrm{R}\sinh\gamma$ and $\mathrm{B} = \mathrm{R}\cosh\gamma$, so $R = -\sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$ **B1 (5)**

(i) $y = a\tanh x + b = \operatorname{sech} x$ **M1**

Thus $a\sinh x + b\cosh x = 1$ **A1**

So $\sqrt{b^2 - a^2}\,\cosh\left(x + \tanh^{-1}\frac{a}{b}\right) = 1$ using first result of question M1

$$\cosh\left(x + \tanh^{-1}\frac{a}{b}\right) = \frac{1}{\sqrt{b^2 - a^2}}$$

$$x + \tanh^{-1}\frac{a}{b} = \pm\cosh^{-1}\left(\frac{1}{\sqrt{b^2 - a^2}}\right)$$

M1

and so

$$x = \pm\cosh^{-1}\left(\frac{1}{\sqrt{b^2 - a^2}}\right) - \tanh^{-1}\frac{a}{b}$$

A1* (5)

(ii)

$$x = \sinh^{-1}\left(\frac{1}{\sqrt{a^2 - b^2}}\right) - \tanh^{-1}\frac{b}{a}$$

M1 A1 (2)

(iii) For intersection to occur at two distinct points, we require two solutions to exist to the equations considered simultaneously. Considering the two graphs, there can be at most only one intersection, which would occur for $x > 0$, if $b \le 0$.

Thus we require $b > a$ and $\left(\frac{1}{\sqrt{b^2-a^2}}\right) > 1$ **M1**

That is $a < b < \sqrt{a^2+1}$. **A1**

Similarly vice versa, if these conditions apply, then there are two solutions and hence two intersections. **E1 (3)**

(iv) To touch, we require two coincident solutions. i.e. $\left(\frac{1}{\sqrt{b^2-a^2}}\right) = 1$

That is $b = \sqrt{a^2+1}$, and equally, if this applies then they will touch, **E1**

so

$$x = -\tanh^{-1}\frac{a}{\sqrt{a^2+1}}$$

M1

and thus $y = a\tanh\left(-\tanh^{-1}\frac{a}{\sqrt{a^2+1}}\right) + \sqrt{a^2+1} = -\frac{a^2}{\sqrt{a^2+1}} + \sqrt{a^2+1} = \frac{1}{\sqrt{a^2+1}}$

A1 **M1** **A1 (5)**

7. If

$$\omega = e^{\frac{2\pi i}{n}}$$

then if $0 \leq r \leq n-1$,

$$(\omega^r)^n = e^{\frac{2\pi irn}{n}} = \left(e^{2\pi i}\right)^r = 1^r = 1$$

M1

So $1, \omega, \omega^2, \dots , \omega^{n-1}$ are the *n* roots of $z^n = 1$, that is of $z^n - 1 = 0$. **A1**

Thus $(z - \omega^r)$ is a factor of $z^n - 1$ **B1**

Hence $z^n - 1 = k(z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$ and comparing coefficients of z^n , $k = 1$

M1

So as required $(z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) = z^n - 1$ **A1* (5)**

(i) Without loss of generality, let X_r be represented by ω^r **M1**

Then P will be represented either by $re^{\frac{\pi i}{n}} = z$, or $re^{\left(\frac{\pi}{n}+\pi\right)i} = z'$ with $|OP| = r$ **M1**

$$|PX_0| \times |PX_1| \times \dots \times |PX_{n-1}| = \left|1 - re^{\frac{\pi i}{n}}\right| \left|\omega - re^{\frac{\pi i}{n}}\right| \dots \left|\omega^{n-1} - re^{\frac{\pi i}{n}}\right|$$

M1

$$= |(z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})| = |z^n - 1| = \left|r^n e^{\pi i} - 1\right| = |-r^n - 1| = r^n + 1$$

A1 **A1** **A1***

or $|z'^n - 1| = \left|r^n e^{(n+1)\pi i} - 1\right| = \left|r^n e^{\pi i} e^{n\pi i} - 1\right| = |-r^n - 1| = r^n + 1$ as $e^{n\pi i} = 1$ because *n* is even. **E1 (7)**

So $|PX_0| \times |PX_1| \times \dots \times |PX_{n-1}| = |OP|^n + 1$ as required.

For *n* odd,

$|PX_0| \times |PX_1| \times \dots \times |PX_{n-1}| = |z^n - 1| = \left|r^n e^{\pi i} - 1\right| = |-r^n - 1| = r^n + 1 = |OP|^n + 1$

M1 A1

or

$$|PX_0| \times |PX_1| \times \dots \times |PX_{n-1}| = \left|z'^n - 1\right| = \left|r^n e^{(n+1)\pi i} - 1\right| = |r^n - 1| = |OP|^n - 1$$

B1

if $|OP| \geq 1$, and $= 1 - |OP|^n$ if $|OP| < 1$ **A1 (4)**

(ii)

$$|X_0X_1| \times |X_0X_2| \times \ldots \times |X_0X_{n-1}| = |(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1})|$$

M1

But

$$(z-1)(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1}) = z^n - 1$$

and so

$$(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1}) = \frac{z^n-1}{z-1} = z^{n-1} + z^{n-2} + \cdots + 1$$

M1

$$(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1}) = z^{n-1} + z^{n-2} + \cdots + 1$$

A1

is true for all z so for $z = 1$, $(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1}) = 1 + 1 + \cdots + 1 = n$ **A1* (4)**

8. (i) $f(x) + (1-x)f(-x) = x^2$

Let $x = -u$, then $f(-u) + (1--u)f(--u) = (-u)^2$

i.e. $f(-u) + (1+u)f(u) = u^2$

Let $u = x$, then $f(-x) + (1+x)f(x) = x^2$ as required. **E1**

Substituting for $f(-x)$ from the equation just obtained in the original, **M1**

$$f(x) + (1-x)\left(x^2 - (1+x)f(x)\right) = x^2$$

Thus $x^2 f(x) = x^3$, and hence $f(x) = x$ **M1 A1**

Verification:- $x + (1-x) \times -x = x - x + x^2 = x^2$ as required. **B1 (5)**

(ii)

$$K\left(K(x)\right) = K\left(\frac{x+1}{x-1}\right) = \frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

M1 **M1** **A1* (3)**

as required.

$$g(x) + xg\left(\frac{x+1}{x-1}\right) = x$$

So

$$g\left(\frac{x+1}{x-1}\right) + \left(\frac{x+1}{x-1}\right)g\left(\frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1}\right) = \left(\frac{x+1}{x-1}\right)$$

M1

That is

$$g\left(\frac{x+1}{x-1}\right) + \left(\frac{x+1}{x-1}\right)g(x) = \left(\frac{x+1}{x-1}\right)$$

A1

So substituting for $g\left(\frac{x+1}{x-1}\right)$ from the equation just obtained in the initial equation **M1**

$$g(x) + x\left(\left(\frac{x+1}{x-1}\right) - \left(\frac{x+1}{x-1}\right)g(x)\right) = x$$

$$[(x-1) - x(x+1)]g(x) + x(x+1) = x(x-1)$$

$$(-x^2 - 1)g(x) = -2x$$

$$g(x) = \frac{2x}{(x^2+1)}$$

M1 A1* (5)

Not required - verification:-

$$\frac{2x}{(x^2+1)} + x\frac{2\left(\frac{x+1}{x-1}\right)}{\left(\left(\frac{x+1}{x-1}\right)^2+1\right)} = \frac{2x}{(x^2+1)} + x\left(\frac{2(x+1)(x-1)}{(x+1)^2+(x-1)^2}\right) = \frac{2x}{(x^2+1)} + x\frac{2(x^2-1)}{2(x^2+1)}$$

$$= \frac{2x + x(x^2-1)}{(x^2+1)} = \frac{x(2+x^2-1)}{(x^2+1)} = x$$

as expected.

(iii)

$$h(x) + h\left(\frac{1}{1-x}\right) = 1 - x - \frac{1}{1-x}$$

(Equation A)

$$h\left(\frac{1}{1-x}\right) + h\left(\frac{1}{1-\left(\frac{1}{1-x}\right)}\right) = 1 - \left(\frac{1}{1-x}\right) - \frac{1}{1-\left(\frac{1}{1-x}\right)}$$

M1 A1

Thus

$$h\left(\frac{1}{1-x}\right) + h\left(\frac{x-1}{x}\right) = 1 - \left(\frac{1}{1-x}\right) + \left(\frac{1-x}{x}\right)$$

(Equation B)

Then

$$h\left(\frac{x-1}{x}\right) + h\left(\frac{1}{1-\left(\frac{x-1}{x}\right)}\right) = 1 - \left(\frac{x-1}{x}\right) - \frac{1}{1-\left(\frac{x-1}{x}\right)}$$

M1 A1

That is

$$h\left(\frac{x-1}{x}\right) + h(x) = 1 - \left(\frac{x-1}{x}\right) - x$$

(Equation C)

A+C-B gives **M1**

$$2h(x) = 1 - x - \frac{1}{1-x} + 1 - \left(\frac{x-1}{x}\right) - x - \left(1 - \left(\frac{1}{1-x}\right) + \left(\frac{1-x}{x}\right)\right)$$

A1

$$2h(x) = 1 - 2x$$

So

$$h(x) = \frac{1}{2} - x$$

A1 (7)

Not required - verification:-

$$\frac{1}{2} - x + \frac{1}{2} - \frac{1}{1-x} = 1 - x - \frac{1}{1-x}$$

as expected.

9. $PX = \frac{2}{3}\sqrt{3}a = \frac{2}{\sqrt{3}}a$ or alternatively $PX = a\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}a$ **M1 A1**

So the extension is $\frac{2}{\sqrt{3}}a - l$. **A1* (3)**

Displacing X a distance x towards P, RX will be $\sqrt{a^2 + \left(\frac{1}{\sqrt{3}}a + x\right)^2}$ **M1 A1**

and thus the tension in RX will be

$$\frac{\lambda}{l}\left(\sqrt{a^2 + \left(\frac{1}{\sqrt{3}}a + x\right)^2} - l\right) = \frac{\lambda}{l}\left(\sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2} - l\right)$$

M1 A1* (4)

The cosine of the angle between RX and PX produced will be

$$\frac{\frac{1}{\sqrt{3}}a + x}{\sqrt{a^2 + \left(\frac{1}{\sqrt{3}}a + x\right)^2}}$$

B1

so the equation of motion for X , resolving in the direction XP is

$$\frac{\lambda}{l}\left(\frac{2}{\sqrt{3}}a - l - x\right) - 2\frac{\lambda}{l}\left(\sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2} - l\right)\frac{\frac{1}{\sqrt{3}}a + x}{\sqrt{a^2 + \left(\frac{1}{\sqrt{3}}a + x\right)^2}} = m\ddot{x}$$

M1 A1 A1 (4)

$$\left(\sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2} - l\right)\frac{\frac{1}{\sqrt{3}}a + x}{\sqrt{a^2 + \left(\frac{1}{\sqrt{3}}a + x\right)^2}} = \frac{1}{\sqrt{3}}a + x - \frac{l\left(\frac{1}{\sqrt{3}}a + x\right)}{\sqrt{a^2 + \left(\frac{1}{\sqrt{3}}a + x\right)^2}}$$

so

$$\frac{\lambda}{l}\left(\frac{2}{\sqrt{3}}a - l - x\right) - 2\frac{\lambda}{l}\left(\sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2} - l\right)\frac{\frac{1}{\sqrt{3}}a + x}{\sqrt{a^2 + \left(\frac{1}{\sqrt{3}}a + x\right)^2}}$$

M1 A1

$$= \left(\frac{2}{\sqrt{3}}a\frac{\lambda}{l} - \lambda - \frac{\lambda}{l}x\right) - \left(\frac{2}{\sqrt{3}}a\frac{\lambda}{l} + \frac{2\lambda}{l}x - \frac{2\lambda\left(\frac{1}{\sqrt{3}}a + x\right)}{\sqrt{a^2 + \left(\frac{1}{\sqrt{3}}a + x\right)^2}}\right)$$

$$= -\lambda - \frac{3\lambda}{l}x + 2\lambda\left(\frac{1}{\sqrt{3}}a + x\right)\left(\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2\right)^{\frac{-1}{2}}$$

$$= -\lambda - \frac{3\lambda}{l}x + 2\lambda\left(\frac{1}{\sqrt{3}}a + x\right)\frac{\sqrt{3}}{2a}\left(1 + \frac{\sqrt{3}}{2}\frac{x}{a} + \frac{3x^2}{4a^2}\right)^{\frac{-1}{2}}$$

A1

$$\approx -\lambda - \frac{3\lambda}{l}x + \lambda\left(\frac{1}{\sqrt{3}}a + x\right)\frac{\sqrt{3}}{a}\left(1 - \frac{\sqrt{3}}{4}\frac{x}{a}\right)$$

M1 A1

$$\approx -\lambda - \frac{3\lambda}{l}x + \lambda + \frac{\sqrt{3}\lambda x}{a} - \frac{\sqrt{3}}{4}\frac{\lambda x}{a}$$

M1

$$= -\frac{3\lambda}{l}x + \frac{3\sqrt{3}}{4}\frac{\lambda x}{a}$$

$$= -\frac{3\lambda}{4la}(4a - \sqrt{3}l)x$$

A1

This is approximately the equation of simple harmonic motion with period

$$\frac{2\pi}{\sqrt{\frac{3\lambda}{4mla}(4a - \sqrt{3}l)}} = 2\pi\sqrt{\frac{4mla}{3(4a - \sqrt{3}l)\lambda}}$$

as required. **M1 A1*(9)**

10. Resolving upwards along a line of greatest slope initially, if the tension in the string is T,

$$T\cos\beta - mg\sin\alpha = m\frac{u^2}{a\cos\beta}$$

M1 **M1 B1 A1 (4)**

Resolving perpendicular to the slope, if the normal contact force is R,

$$R + T\sin\beta - mg\cos\alpha = 0$$

M1 A1 (2)

The particle will not immediately leave the plane if $R > 0$. **M1**

This is

$$mg\cos\alpha > T\sin\beta$$

A1

So

$$mg\cos\alpha > \frac{m\frac{u^2}{a\cos\beta} + mg\sin\alpha}{\cos\beta}\sin\beta$$

M1

That is

$$g\cos\alpha\cos\beta > g\sin\alpha\sin\beta + \frac{u^2}{a}\tan\beta$$

which becomes $ag(\cos\alpha\cos\beta - \sin\alpha\sin\beta) > u^2\tan\beta$ **M1**

or, as required, $ag\cos(\alpha+\beta) > u^2\tan\beta$ **A1* (5)**

A necessary condition for the particle to perform a complete circle whilst in contact with the plane is that the string remains in tension when the particle is at its highest point in the motion. **E1**

If the speed of the particle at that moment is v , then conserving energy,

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg2a\cos\beta\sin\alpha$$

M1 A1

and thus $v^2 = u^2 - 4ag\cos\beta\sin\alpha$

Resolving downwards along a line of greatest slope, if the tension in the string is now T',

$$T'\cos\beta + mg\sin\alpha = m\frac{v^2}{a\cos\beta}$$

B1

$$T' > 0 \Rightarrow m\left(\frac{v^2}{a\cos\beta} - g\sin\alpha\right) > 0$$

M1

which means that

$$\frac{u^2 - 4ag\cos\beta\sin\alpha}{a\cos\beta} - g\sin\alpha > 0$$

Thus $u^2 > 5ag\cos\beta\sin\alpha$ **A1 (6)**

As we already have $ag(\cos\alpha\cos\beta - \sin\alpha\sin\beta) > u^2\tan\beta$

$$5ag\cos\beta\sin\alpha\tan\beta < ag(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

M1

So $5\sin\alpha\sin\beta < \cos\alpha\cos\beta - \sin\alpha\sin\beta$

i.e. $6\sin\alpha\sin\beta < \cos\alpha\cos\beta$ or, as is required, $6\tan\alpha\tan\beta < 1$ **M1 A1* (3)**

11. (i) Suppose $R = kv$ for some constant k

Then as $\frac{P}{v} - R = ma$, $\frac{P}{4U} - 4kU = 0$ giving $k = \frac{P}{16U^2}$ **B1**

As $ma = \frac{P}{v} - R$, $mv\frac{dv}{dx} = \frac{P}{v} - kv$ **M1**

Separating variables,

$$\int \frac{mv^2}{P - kv^2} dv = \int dx$$

M1

So

$$\frac{m}{P}\int \frac{16U^2v^2}{16U^2 - v^2} dv = \int dx$$

$$\frac{16U^2v^2}{16U^2 - v^2} = 16U^2\left(\frac{16U^2}{16U^2 - v^2} - 1\right) = 16U^2\left(\frac{2U}{4U - v} + \frac{2U}{4U + v} - 1\right)$$

M1 A1

So

$$\left[16U^2\frac{m}{P}(-2U\ln(4U - v) + 2U\ln(4U + v) - v)\right]_U^{2U} = X_1$$

M1 A1

$$X_1 = \frac{16mU^3}{P}(-2\ln 2U + 2\ln 6U - 2 + 2\ln 3U - 2\ln 5U + 1)$$

Thus

$$\lambda X_1 = 2\ln\left(\frac{6U \times 3U}{2U \times 5U}\right) - 1 = 2\ln\frac{9}{5} - 1$$

M1 A1 (9)

(ii) Suppose $R = \mu v^2$ for some constant μ

Then $\frac{P}{4U} - 16\mu U^2 = 0$ giving $\mu = \frac{P}{64U^3}$ **B1**

Again, as $ma = \frac{P}{v} - R$,

$$mv\frac{dv}{dx} = \frac{P}{v} - \mu v^2 = \frac{P - \mu v^3}{v}$$

$$\int \frac{mv^2}{P - \mu v^3} dv = \int dx$$

M1

So

$$\left[\frac{-m}{3\mu}\ln(P-\mu v^3)\right]_U^{2U} = X_2$$

M1 A1

$$X_2 = \frac{-64U^3m}{3P}\left(\ln\frac{7}{8}P - \ln\frac{63}{64}P\right) = \frac{-64U^3m}{3P}\ln\frac{8}{9}$$

Thus $\lambda X_2 = \frac{4}{3}\ln\frac{9}{8}$ **M1 A1 (6)**

(iii) $\lambda X_1 - \lambda X_2 = 2\ln\frac{9}{5} - 1 - \frac{4}{3}\ln\frac{9}{8} = 4\ln 3 - 2\ln 5 - 1 - \frac{8}{3}\ln 3 + 4\ln 2$

M1

$$= \frac{4}{3}\ln 24 - 2\ln 5 - 1 > \frac{1}{3}(4\times 3.17 - 6\times 1.61 - 3) = \frac{1}{3}(12.68 - 9.66 - 3) > 0$$

A1 **M1** **A1**

So X_1 is larger than X_2 **A1 (5)**

12. (i) $X \sim B(100n, 0.2)$ **B1**

So $\mu = 100n \times 0.2 = 20n$ **M1 A1** and $\sigma^2 = 100n \times 0.2 \times 0.8 = 16n$ **M1 A1**

So $P(16n \leq X \leq 24n) = P(|X - 20n| \leq 4n) = P\left(|X - 20n| \leq \sqrt{n} \times \sqrt{16n}\right)$

M1 **M1 A1**

So by Chebyshev, $P(16n \leq X \leq 24n) \geq 1 - \left(\frac{1}{\sqrt{n}}\right)^2 = 1 - \frac{1}{n}$ as required. **A1* (9)**

(ii) Suppose $X \sim Po(n)$ **B1**

Then $\mu = n$ **B1** and $\sigma^2 = n$ **B1**

By Chebyshev, $P(|X - \mu| > k\sigma) \leq \frac{1}{k^2}$

so let $k = \sqrt{n}$ and hence $P(|X - n| > n) \leq \frac{1}{n}$

M1 **A1**

$$P(|X - n| > n) = P(X < 0 \text{ or} X > 2n) = P(X > 2n) = 1 - e^{-n} - ne^{-n} - \frac{n^2 e^{-n}}{2!} - \cdots - \frac{n^{2n} e^{-n}}{2n!}$$

M1 **A1** **A1**

So $1 - e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{2n!}\right) \leq \frac{1}{n}$ **M1 A1**

and hence $1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{2n!} \geq \left(1 - \frac{1}{n}\right)e^n$ **A1* (11)**

13. Let $Y = X - a$, then $\mu_Y = E(Y) = E(X-a) = E(X) - a = \mu - a$ **B1**

$$E((Y-\mu_Y)^4) = E((X-a-\mu+a)^4) = E((X-\mu)^4)$$

B1

$$\sigma_Y{}^2 = E((Y-\mu_Y)^2) = E((X-a-\mu+a)^2) = E((X-\mu)^2) = \sigma^2$$

B1

so the kurtosis of $X - a$ is

$$\frac{E((Y-\mu_Y)^4)}{\sigma_Y{}^4} - 3 = \frac{E((X-\mu)^4)}{\sigma^4} - 3$$

which is the same as that for X **B1* (4)**

(i) If $X \sim N(0, \sigma^2)$ then it has pdf

$$\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2}$$

So

$$E((X-\mu)^4) = \int_{-\infty}^{\infty} x^4 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} x^3 x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

M1 A1

By parts,

$$\int_{-\infty}^{\infty} x^3 x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2} dx = \left[x^3 \times -\sigma^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 3x^2 \times -\sigma^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

M1 A1

$$= 0 + 3\sigma^2\sigma^2 = 3\sigma^4$$

A1

So the kurtosis is

$$\frac{3\sigma^4}{\sigma^4} - 3 = 0$$

as required. **(5)**

(ii)

$$T^4 = \left(\sum_{r=1}^{n} Y_r\right)^4 = \sum\left(Y_r{}^4 + 4Y_r{}^3Y_s + 6Y_r{}^2Y_s{}^2 + 12Y_sY_tY_r{}^2 + 24Y_rY_sY_tY_u\right)$$

M1 A1

where the summation is over all values without repetition.
As the Ys are independent, the expectation of products are products of expectations and as

$E(Y) = 0$,

$$E(T^4) = E\left(\sum\left(Y_r{}^4 + 6Y_r{}^2Y_s{}^2\right)\right) = E\left(\sum_{r=1}^{n} Y_r{}^4\right) + E\left(\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} 6Y_r{}^2Y_s{}^2\right)$$

M1

$$= \sum_{r=1}^{n} E\left(Y_r{}^4\right) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} E\left(Y_r{}^2\right)E\left(Y_s{}^2\right)$$

A1* (4)

(iii)

$$\frac{E((X_i-\mu)^4)}{\sigma^4} - 3 = \kappa$$

Let $Y_i = X_i - \mu$ then by the first result, the kurtosis of Y_i is ,

i.e.

$$\frac{E\left(Y_i{}^4\right)}{\sigma^4} - 3 = \kappa$$

so $E\left(Y_i{}^4\right) = (3+\kappa)\sigma^4$ **M1 A1**

$$E\left(\sum_{i=1}^{n} X_i\right) = n\mu$$

and

$$Var\left(\sum_{i=1}^{n} X_i\right) = n\sigma^2$$

B1

so the kurtosis of

$$\sum_{i=1}^{n} X_i$$

is

$$\frac{E((\sum_{i=1}^{n} X_i - n\mu)^4)}{(n\sigma^2)^2} - 3 = \frac{E((\sum_{i=1}^{n} Y_i)^4)}{n^2\sigma^4} - 3$$

M1

Let

$$T = \sum_{r=1}^{n} Y_r$$

Then we require

$$\frac{E(T^4)}{n^2\sigma^4} - 3$$

which by (ii) is

$$\frac{\sum_{r=1}^{n} E\left({Y_r}^4\right) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} E\left({Y_r}^2\right)E\left({Y_s}^2\right)}{n^2\sigma^4} - 3$$

M1 A1

$$= \frac{n(3+\kappa)\sigma^4 + 3n(n-1)\sigma^2\sigma^2}{n^2\sigma^4} - 3 = \frac{3+\kappa+3(n-1)-3n}{n} = \frac{\kappa}{n}$$

A1* (7)